International Journal of Innovation in Science and Mathematics
Volume 6, Issue 2, ISSN (Online): 2347–9051

Infinity Paradoxes in the Light of its More Accurate Algebraic Representation

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Date of publication (dd/mm/yyyy): 28/04/2018

***Abstract* – Even though having the same cardinality, infinities are not all identical.**

In this paper it will be shown that a more opportune algebraic representation of infinity can complete the perspective of it and, in some cases, can give us a new sight of some paradoxes, as in the case of the Hilbert's Hotel.

***Keywords* – Infinity, Algebraic Representation of Infinity, Cantor's Diagonal Method, De Zolt's Axiom, Hilbert's Hotel.**

I. Introduction

Cantor's concept of equipotence for infinite sets (1891) can be thought to be in contradiction with De Zolt's axiom on comparison among areas (1881). After all, even though having the same cardinality, infinities are not all identical.

In this paper it will be shown that a more opportune algebraic representation of infinity can complete the perspective of it and, in some cases, could give us a new sight of some paradoxes, as in the case of the Hilbert's Hotel (2013 [1917-1933])[1].

II. About Infinity and its Symbolic Representation

An infinite set can be put in biunivocal correspondence with its proper infinite subsets: for example the infinities of even (E) and odd (O) numbers and the infinity of natural numbers (N) are equipotent although in any range of natural numbers $N \approx 2\ E$ and $N \approx 2\ O$. Furthermore no number can modify infinity when added to it or subtracted from it. Assumed $n \neq 0$:

$$n + \infty = \infty. \tag{1}$$

Even $\infty + \infty = \infty$ and $\infty\ \infty = \infty$ and it is not $\infty + \infty = 2\ \infty$ or $\infty\ \infty = \infty^2$.

However subtracting ∞ from both the members of **1**, it is $n = 0$, hence contradicting the assumption $n \neq 0$. The explanation of this apparent absurdity is that ∞ is not actually a number and then no number can modify it.

In our opinion, on the contrary, numbers can be added to (or subtracted from) ∞, changing its algebraic representation while not changing its nature of infinity. Indeed we can build up a set, say it S, as the union of a subset S_A containing infinite elements (say them a_1, a_2, and so on) which can be associated with natural numbers and another subset (say it S_B) containing the element to be added (say this element b_1). Then it is here affirmed that:

$$\infty + n = \infty\ + n \tag{2}$$

and not ∞. In the same way n elements can be subtracted from an infinite set S and this will be still infinite obviously without these elements thus being well represented by ∞ - n. This solution may seem trivial and also the result of pedantry but it implies that there is no substantial difference between ordinal and cardinal arithmetic when talking about infinity: a new element added to a set of infinite elements does not vanish into it. In the same way an element in a set made of only one element does not vanish if infinite elements are added to it. To make some examples, the thesis according to which if infinite elements are taken twice two infinities are obtained, whilst if infinite pairs of elements are taken only one infinite is obtained (Cheng, 2018 [2017], pp. 157-160), is not correct. Taking infinite pairs actually means associating each pair with a natural number, thus there are not infinite elements, but infinite pairs of elements: $2\ \infty$ and not ∞, unless each element is alternately associated with natural numbers.

If odd numbers only or even numbers only are individually taken, they are actually taken infinite times but they are taken at the rate of one element for each couple of natural numbers. Therefore, assumed $N = \infty$, even though O and E have the same cardinality of N, we can more correctly write: $O = (1\ /\ 2)\ \infty$, or $E = (1\ /\ 2)\ \infty$.

In summary assuming that $N = \infty$ the other equipotent sets or subsets can be indicated by the symbol ∞, but preceded by a coefficient and followed (or preceded), when required, by another arithmetical operator.

III. The Diagonal Method

By means of the diagonal method Cantor (1891) shows that rational numbers (among other sets of numbers) are equivalent to natural numbers.

After all already Galilei[2] noted that natural numbers and their squares can be put in biunivocal correspondence.

Cantor's diagonal method is almost universally accepted, despite some scholars' dissenting opinions (Zenkin, 2004; Zhuang[3]).

However here we want to stress that the diagonal method and, more generally, the procedure of putting the elements of two sets in biunivocal correspondence, are processes and a process has sense only if the time factor is considered, since every process has a time axis (we are then in the field of potential infinities).

In the case of the correspondence between numbers and

1 About the story of Hilbert's hotel, see Kragh (2014).

2 See Zichichi, 2009, p. 217.

3 Available: https://philarchive.org/archive/ZHUWAO .

their squares the latter are grater and greater than the former, thus being more and more distant on the straight line representing them. The lengthening of the distance can be related to the time required to realize the linking: it tends to infinity while n grows, thus this process has not an end.

The same happens in the case of the diagonal method.

In the same way the Hilbert's Hotel can not host infinite new guests: the distance to travel (and the necessary time) to reach the new rooms, growing n, tends to ∞, and then the process can not end.

As regard to the case of n new guests, on the contrary, the situation looks like the one described by **2**: n elements are added to an infinity and the output is believed to be $\infty + n$. The infinite rooms of the Hilbert's Hotel can be represented on a straight line, more precisely on a half-line. We can fix the coordinate 0 everywhere on a straight line: the two halves of the straight line will be equivalent, but the fixation of the zero point is a creative act of a mathematical reality: from that moment it is no longer correct to change the coordinates of the straight line. Even admitting that the half-line can be moved to the right, things do not change: we have an infinity plus n, as in **2**.

IV. The Continuum

Let us focus now on the set of real numbers R and represent it on a straight line. Assuming

$$\infty + \infty = \infty \qquad (3)$$

means overlapping two straight lines (or better two half-lines), in fact wiping one off. The points on each straight line, that is their coordinates, are infinite and their length is zero but they are not nothing: they exist as coordinates. The overlap results in the elimination of the overlapped point. When two infinities, in this case two straight lines, are added, they are not overlapped. Furthermore when two straight lines are added, they are not put in a string: an infinity (straight line) cannot follow another infinity (straight line) because it does not have an end. In the same way, also in this case, $\infty + 1$ is not equal to ∞, but it is:

$$\infty + 1 = \infty + 1 \qquad (4)$$

i. e. a set with two subsets, the first with a straight line and the second with a line segment whose length is 1. Also adding a point (say it $1\ (0)$, since its length is 0[4]), there will be two sets (or subsets), the first with a straight line and the second with a point.

Moreover, n infinities can be represented by n parallel straight lines (otherwise at least $n - 1$ points are in commune). On a plane infinite straight lines exist and in the space there are infinite planes: therefore, even though all are equipotent infinities, they are not identical. A line segment can be put infinite times in a straight line as a straight line can be put in a plane and a plane in a space.

In summary all these sets, according to the Cantor's approach, have the same cardinality, but by means of a more opportune algebraic representation, into this single infinity level we can distinguish sub-levels of infinity. The ratio between them, even if they are equivalent in the Cantor's frame, can match infinity. Obviously the same happens in the other levels of infinity.

V. Conclusion

In this paper we discussed about the algebraic representation of infinities. We highlighted how sometimes this more accurate representation, taking into account the time axis too, is useful to understand the relation between infinite sets and their proper infinite subsets. It can cast new light on apparently paradoxical situations like the Hilbert's Hotel.

With this approach we do not want to criticize the diagonal argument of Cantor, but to introduce a further, more detailed classification of the infinities.

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4 See Paolilli (2017).

Since it is clear that we may have one line segment longer than another, each containing an infinite number of points, we are forced to admit that, within one and the same class, we may have something greater than infinity, because the infinity of points in the long line segment is greater than the infinity of points in the short line segment.Â Galileoâ€™s Dialogo â€¢ Salviati: If I should ask further how many squares there are, one might reply truly that there are as many as the corresponding number of roots, since every square has its own root and every root its own square, while no square has more than one root and no root more than one square. â€¢Â Yet at the outset we said there are many more numbers than squares, since the larger portion of them are not squares.