



STUDIES IN LOGIC

AND

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S. ABRAMSKY / S. ARTEMOV / D.M. GABBAY / A. KECHRIS / A. PILLAY / R.A. SHORE

EDITORS

*Many-Dimensional
Modal Logics:
Theory and
Applications*

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ELSEVIER

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MANY-DIMENSIONAL MODAL LOGICS: THEORY AND APPLICATIONS

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**With love and gratitude to our parents
this book is dedicated**

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'You've got to learn to think multi-dimensionally. . .
If you'd like to know, I can tell you that in your universe you move freely in three dimensions that you call space. You move in a straight line in a fourth, which you call time, and stay rooted to one place in a fifth, which is the first fundamental of probability. After that it gets a bit complicated, and there's all sorts of stuff going on in dimensions 13 to 22 that you really wouldn't want to know about.'

Douglas Adams 'Mostly Harmless'

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Preface

Modal logic is a discipline of many facets. It was baptized in philosophy, and for a long time it was known as ‘the logic of necessity and possibility.’ The modal analysis of the ‘mathematical necessity’—provability—brought modal logic to the foundations of mathematics. The discovery of topological and algebraic semantics for modal logic connected it with general topology and universal algebra, and the fact that first-order logic can be regarded as a propositional modal logic opened a ‘modal perspective’ in classical mathematical logic. But the most amazing metamorphosis happened when it turned out that modal logic could provide languages for talking about various relational structures, such as state transition systems for computer programs, semantic networks for knowledge representation, or attribute value structures in linguistics—languages that combined both sufficient expressive power and effectiveness! This opened new rich and rapidly growing application fields in computer science, artificial intelligence, linguistics, as well as in mathematics itself.

A great many systems with various kinds of modal operators have been constructed in the last few decades in order to provide effective formalisms for talking about time, space, knowledge, beliefs, actions, obligations, etc.: temporal, spatial, epistemic, dynamic, deontic, and so forth. However, modern applications often require rather complex formal models and corresponding languages that are capable of reflecting different features of the application domain. For instance, to analyze the behavior of a multi-agent distributed system we may need a formalism containing both epistemic operators for capturing the knowledge of agents and temporal operators for taking care of the development of this knowledge in time. In other words, we should construct a suitable *combination* of epistemic and temporal logics. Borrowing the geometrical terminology, one can call the resulting hybrid a *many-dimensional modal logic* (later on we shall see that this name is not merely a nice metaphor).

The algorithmic properties of such many-dimensional hybrids that appeared more or less independently and with different motivations in computer science, artificial intelligence, algebraic logic, and modal logic, turn out

to be quite different from those of their well-known and well-behaved one-dimensional components. In particular, the complexity of decision algorithms may increase dramatically, even up to undecidability; two fairly simple finitely axiomatizable systems may give rise to a hybrid which is not even recursively enumerable, etc.

To study the computational behavior of many-dimensional modal logics is the main aim of this book. More precisely, as suggested by its title, our aim is twofold. On the one hand, we are concerned with providing a solid mathematical foundation for the discipline characterized in (Blackburn *et al.* 2001) as

... a branch of modal logic dealing with special relational structures in which the states, rather than being abstract entities, have some inner structure. More specifically, these states are tuples or sequences over some base set ... Furthermore, the accessibility relations between these states are (partly) determined by this inner structure of the states.

On the other hand, we show that many seemingly different applied many-dimensional systems (e.g., multi-agent systems, description logics with epistemic, temporal and dynamic operators, spatio-temporal logics, (fragments of) first-order temporal or epistemic logics, etc.) fit in perfectly with this theoretical framework, and so their computational behavior can be analyzed using the developed machinery. Thus, we contribute to filling in the gap between the mathematical theory of modal logic and applications in computer science and artificial intelligence, which were developing in parallel, often independently of each other (witness description logics created in the field of knowledge representation and proved to be terminological variants of well-known modal logics). This gap is clearly reflected in the existing literature. Take, for example, two recent books: (Fagin *et al.* 1995) is an excellent exposition of applied epistemic logic, but it avoids difficult proofs, say, the complexity results are only formulated; the first monograph on multi-dimensional modal logics (Marx and Venema 1997), on the contrary, considers mostly mathematical aspects of modal systems originating in algebraic logic.

This book also reflects a new direction in applied logic in general and in modal logic in particular that has become apparent in the last few years: we mean the direction towards constructing and investigating complex combined systems out of relatively simple ones. It has manifested itself in a number of international conferences (e.g., 'Frontiers of Combining Systems' FroCoS'96–FroCoS'02) and subsequent volumes (Baader and Schulz 1996, de Rijke and Gabbay 2000, Kirchner and Ringeissen 2000, Armando 2002), special issues of *Notre Dame Journal of Formal Logic* (de Rijke and Blackburn 1996) and *Studia Logica* (Gabbay and Pirri 1997, Kurucz *et al.* 2002), and the monograph (Gabbay 1999).

Complex logical systems and knowledge representation formalisms present many challenging problems for investigation. In this book we concentrate on three of them that are regarded as fundamental in both mathematical logic and theoretical computer science. These are:

- The *decision problem*, i.e., to find out whether there exists an algorithm that is capable of deciding a given reasoning task (say, satisfiability or entailment) for a logic.
- The *complexity problem*, i.e., to find lower and upper bounds for the computational complexity of a possible algorithmic solution to the decision problem.
- The *axiomatization problem*, i.e., to give an effective (preferably finite) syntactical characterization of a semantically defined logic, or to provide an adequate semantics for a syntactically defined one.

The first two problems are concerned with *effectiveness* (or programmability) and *efficiency* (or fast programmability) of logical systems, which make them fundamental in artificial intelligence and other practical fields of computer science as well. The third problem is connected with the proof-theoretic approach: it can be understood as describing the most essential features of a logic starting from which all others are derivable. Thus, the axiomatization problem underpins possible implementations of decision procedures.

The direct *practical* use of decidability, complexity and axiomatizability results may be not that obvious.

'Negative' results, say undecidability, are clearly useful: they warn us not to waste time with implementing this or that 'decision' algorithm. These results lead to a new research programme: (1) to find semi-decision (i.e., sound but incomplete) procedures, (2) to search for decidable fragments of the logic in question, (3) to modify the logic by making it decidable, etc. Similarly, a result establishing a high computational complexity or non-axiomatizability may force the researcher to devise another language to model her application domain.

On the other hand, a positive decidability result does not yet guarantee that trying to implement a decision procedure is not a waste of time: after all the British Museum algorithm is also a decision procedure. It may seem that the only use of a positive solution to the decision problem is the conclusion that it is not provable (with the existing concepts in recursion and complexity theory) that implementing a decision procedure is hopeless.¹ This interpret-

¹Even this cautious claim may appear too strong. Here, for example, is a citation from (Johnson 1990): 'Even those unimpressed with the difficulty of problems in 2- or 3-EXPTIME will have to admit that if a problem is decidable but not in ELEMENTARY, it might as well not be decidable at all.' So, according to Johnson, there are concepts in com-

ation of decidability results can hardly justify spending an enormous amount of energy for obtaining them.

Actually, it is not only the result itself, not only the first ‘sanity check’ of the system, not only an indication of how to implement a decision procedure that motivates the researcher, but also a deep insight into properties of the logical system that can be extracted from decidability and axiomatizability *proofs*.² It is the *proof* of a complexity result that provides the researcher with insights into the sources of high computational costs and can be used to guide the search for more efficient languages—see (Gurevich 1990, 1995) for a similar position. To conclude this discussion: we believe that the decidability, complexity, and axiomatizability theorems we present in this book are of interest mainly because their proofs provide a better understanding of the logical formalisms we consider and, at the same time, are often useful for the design of practical procedures (to a certain extent, this is illustrated in Chapter 15 which presents tableau calculi for modal description logics).

The book is organized in the following way. Part I may serve as an easy introduction to modal logic and its applications. Chapters 1 and 2 introduce the basic modal logics we deal with, explain their roots, motivations, syntax, semantics and application fields. At the end of Chapter 2 we show useful ‘semantical level’ reductions between many of these logics. In Chapter 3, we consider a number of many-dimensional systems constructed in logic, artificial intelligence and computer science, and establish connections between the various formalisms.

Part II is the technical core of the book. Here we develop a mathematical theory for handling a spectrum of ‘abstract’ combinations of modal logics, ranging from fusions (known also as independent joins or dovetailings), in which the modal operators of different components do not interact at all, to products of modal logics, where such an interaction is rather strong. The ideas, tools and techniques developed in Part II as well as the obtained results will be used in the subsequent two parts to investigate the computational behavior of first-order modal and temporal logics and some combined knowledge representation formalisms.

In Part III we consider first-order modal and temporal logics in the two-

plexity theory which allow us to classify certain decidable problems as ‘non-implementable’ ones. We do not agree with this for the simple reason that a non-elementary problem may be solvable in polynomial time in all *practical cases*. For certain highly complex problems a useful and *complete* decision algorithm may exist; see e.g. (Horrocks *et al.* 1999, Hustadt and Schmidt 2000). Klarlund *et al.* (2002), writing about the prover MONA in monadic second-order logic, observe that ‘perhaps surprisingly, this [NONELEMENTARY] complexity also contributes to successful applications, since it is provably linked to the succinctness of the logic.’

²That is why decidability results obtained by means of embeddings into extremely expressive formalisms (like monadic second-order logics $S1S$ or $S\omega S$) may appear somewhat disappointing.

dimensional perspective. It has been known since the 1960s that it is extremely hard to deal with these logics (and that they can be very important for applications). But in contrast to classical predicate logic, where the early undecidability results of Turing and Church stimulated research and led to a rich and profound theory concerned with classifying fragments of first-order logic according to their decidability (see, e.g., Börger *et al.* 1997), there were no serious attempts to convert the ‘negative’ results in first-order modal and temporal logic into a classification problem. Apparently, the extremely weak expressive power of the modal and temporal formulas required to prove undecidability left no hope that any useful decidable fragments located ‘between’ propositional and first-order modal and temporal logics could ever be found. However, the studies of many-dimensional propositional modal logics have brought a new insight into the first-order case. In Part III we present a number of recent results concerning decidable and axiomatizable fragments of various first-order modal and temporal logics, and try to draw a borderline between ‘the decidable’ and ‘the undecidable.’

Part IV applies the developed techniques and obtained results to analyze the computational behavior of three kinds of knowledge representation formalisms: temporal epistemic logics, description logics with modal and temporal operators, and spatio-temporal logics. In particular, we show how the method of quasimodels, developed for proving decidability, can be used for devising tableau decision procedures for some of these logics.

The genre of the book can be defined as a research monograph. It brings the reader to the front line of current research in the field by showing both recent achievements and directions of future investigations (in particular, multiple open problems). On the other hand, well-known results from modal and first-order logic are formulated without proofs and supplied with references to accessible sources.

The intended audience of this book is primarily those researchers who use logic in computer science and artificial intelligence. More specific areas are, e.g., knowledge representation and reasoning, in particular, terminological, temporal and spatial reasoning, or reasoning about agents. For ‘pure’ logicians Parts II and III may be of chief interest. Logicians looking for possible applications may find some useful ideas in Parts I and IV. And we also believe that researchers from certain other disciplines, say, temporal and spatial databases or geographical information systems, will benefit from this book as well.

We conclude this preface by putting the subject of the book into a more general perspective of ongoing and future research. As we have said above, many-dimensional modal logics are just one example of ‘combined logical systems.’ And even within this smaller area there may be different ways of constructing complex logics out of relatively simple ones. Our approach is

basically semantical: given two (or more) classes of ‘one-dimensional’ Kripke frames characterizing logics L_i , we construct a class of two- (or higher-) dimensional Kripke structures reflecting some desirable features of the target combination of the L_i , and then investigate the logic determined by this class. However, logics do not always come equipped with their Kripke semantics. They may be given by some kind of algebraic structures or purely syntactically, as Hilbert-, Gentzen-, tableau-, resolution-, etc. style calculi. Thus, we need a spectrum of *methodologies* providing us with means of combining homogeneously given logics (say, tableau systems) and perhaps meta-methodologies for combining methodologies. These challenging problems are far beyond the scope of this book; many of them are still open for investigation.

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