THE EVOLUTION OF NUMBER IN MATHEMATICS

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Abstract. Mathematics is the science of *number*, *time*, *motion*, and *space* (FAQ1). Over the past five millenia* our understanding of these four concepts and their inter-relations has changed significantly, due largely to mathematical researches which over the same period have transformed mathematics and its applications. Where the four were once viewed as distinct concepts,[1] they are now unified in a generalized concept of number that has been enlarged through algebraic and analytic extension, in a process that characterizes modern mathematics (FAQ2). In this paper we trace the evolution of number from the whole numbers known since the dawn of counting, to the discovery of the octonions in the 19th century and their connection with string theory and the grand unified 'theory of everything' in the 20th century.(FAQ13) The main paper is a very brief three pages, followed by remarks, historical notes, FAQs, and appendices covering theorems with proofs and an annotated bibliography. A project guide is in progress for learning the material through exploration and self-discovery.

For scholars and laymen alike it is not philosophy but active experience in mathematics itself that can alone answer the question: 'What is Mathematics?'
Richard Courant (1941), book of the same title

"One of the disappointments experienced by most mathematics students is that they never get a course on mathematics. They get courses in calculus, algebra, topology, and so on, but the division of labor in teaching seems to prevent these different topics from being combined into a whole. In fact, some of the most important and natural questions are stifled because they fall on the wrong side of topic boundary lines. Algebraists do not discuss the fundamental theory of algebra because "that's analysis", and analysts do not discuss Riemann surfaces because "that's topology," for example. Thus, if students are to feel they really know mathematics by the time they graduate, there is a need to unify the subject."
John Stillwell (1989), Mathematics and Its History

Number. refers to quantities which can be combined through computation using one or more operations.[2] Whole numbers are familiar to all who have experience with counting,* integers to those who have engaged in practical commerce (trade and exchange),† and rational numbers from surveying and measurement, or from the arithmetic and algebra taught in schools. The essential properties of these number collections have been extracted into a generalized algebraic setting, e.g. semi-rings generalize the positive whole numbers $\mathbb{N}^+$, monoids the integers $\mathbb{Z}$, and rings the rationals $\mathbb{Q}$. These abstract structures provide an axiomatic framework for computing with quite non-numerical objects.(FAQ4). As an extreme example, Rock-paper-scissors is a finite magma, closed, commutative, non-associative, having neither inverses nor identity elements, but within which computations follow clear rules understandable even by school age children.[3]

The rational numbers are in theory sufficient for all of engineering and applied science, since every measured number has finite precision and therefore must itself be rational.[4] But the rationals are an infinite set (FAQ5), so no finite precision computing machine could represent them all.[5] As such, computational mathematics uses the large but *finite* set of floating point numbers. These are able to represent both very large and very small numbers with a known maximum error for numbers falling within range.[6]

The pressure to extend the exact numbers (those having no error) beyond the rationals comes from 1) numerical mathematics via error analysis, 2) geometry via incommensurability of length and area, 3) algebra via solving algebraic polynomials and the desire for algebraic closure, and 4) analysis via the continuum, completeness (freedom from 'holes') and connectedness (continuous paths connecting any two points). This, as we shall see, takes us as far as the one-dimensional reals $\mathbb{R}$ and two-dimensional complex numbers $\mathbb{C}$ (to be explained further on). From here, it is sheer curiosity, the fanciful exploration of a 'what

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*Extended anthropological studies of the indiginous Piraha people of the Brazilian Amazon have shown that counting is *not* universal amongst all human cultures, as was previously thought (FAQ8)

†Counting tokens, abacus, knots in quipo, tally sticks, beans in gourds, have all been used to maintain commercial records and execute transactions. Borrowing and lending motivated the use of negative numbers.

if...' and a 'why not...' that culminates in the four-dimensional quaternions $\mathbb{H}$ and eight-dimensional $\mathbb{O}$. But as has often happened in mathematics, what starts off as fancy typically finds an application. The octonions, discovered in the 19th century, earned their place in the 20th century through the deep mathematics that explains why this tower of algebras is the way it is, why it stops at 8 dimensions (FAQ13), and now, in the 21st century the possibility that these 8-dimensional numbers may actually be the best language for describing the fundamental 'grand unified theories' of the universe. But first things first.

Computationally, exact numbers are required to improve floating point numerical algorithms and identify the best arrangement of calculations to control the error that accumulates during extended computations.[7]

But geometry shows that the rational exact numbers are not plentiful enough to include many quantities which are undeniably qualified to be considered as 'number', despite there being infinitely many rationals, both at the large and small scales, and the property of Archimedes which says between any two rationals there is always another. In on other words, there is no smallest quantum of granularity between rational numbers. The existence of demonstrably irrational numbers should come as a shock—as it was for the Greeks—but once one is found, then there are a whole lot more: these appear commonly as geometric lengths (e.g. diagonals of squares and cubes e.g. $\sqrt{2}$ and $\sqrt[3]{2}$), ratios of lengths (e.g. circumference to diameter π, and the golden mean), chords of circles ($2\sin\theta$ for rational θ, i.e. not multiples of π), arclengths of ellipses,[8] rates (e.g. growth rate e under continuous compounding), or as binary decimal expansions whose digits encode a parameterized decision problem (e.g. setting the nth binary digit to 1 if the nth integer is prime, else 0) All of these irrational quantities exist in the sense that they can be defined precisely and computed to arbitrary precision using rational numbers (typically using iteration) even though they themselves are demonstrably not rational (FAQ9).

Algebraically, while the rationals are closed with respect to arithmetic–plus, minus, multiply, divide– they are not closed with respect to algebraic operations (root, power). For example, it is not possible to solve, within the rationals, all algebraic equations having rational coefficients. For example $x^2 - c = 0$ has no rational solutions either when $c < 0$ or c is prime, since $\mathbb{Q}$ contains neither $\sqrt{-1} = i$ nor $\sqrt{p}$ for any prime p.[9]

The simplest expansion is by field extension $\mathbb{Q}[F]$, where F is the set of constants required to keep the system closed, e.g. $\mathbb{Q}[i := \sqrt{-1}]$ or $\mathbb{Q}[\{\sqrt[3]{2}, \sqrt[3]{2}^2\}]$. Expanding the rational numbers $\mathbb{Q}$ to include these 'new' numbers requires defining what addition and multiplication look like so as to 1) preserve the relations of the existing numbers, and 2) ensure that what's added preserves arithmetic closure.[10] Multiplication within the algebraically extended system is defined by treating each number as a binomial in the algebraic constants $x \in F$.

Listing the constructions adding numbers to $\mathbb{Q}$, we have:

(1) finite field extensions, e.g. by $\sqrt{2}$, $\{\sqrt[3]{2}, \sqrt[3]{2}^2\}$, i, or indeed a fixed literal x;
(2) constructible numbers: intersections obtained through a finite number of operations with a straight-edge and compass, e.g. $\sqrt{2}$;
(3) algebraic numbers: all solutions of polynomials with rational coefficients—note this includes $i = \sqrt{-1}$, $\sqrt[3]{2}$ ($x^3 - 2 = 0$);
(4) periodic numbers: integrals of algebraic functions which includes non-algebraic transcendental numbers such as arc-length of an ellipse (elliptic integrals[**?**], [**?**]);
(5) computable numbers: for which a finite terminating algorithm can be given for calculating the number to arbitrary precision, e.g. π, $e^r = \lim_{n\to infty}(1 + r/n)^n$, $\sqrt{\pi}$, e^π; [11] and
(6) definable numbers: any numbers which can be defined using first order logic (arithemtically definable) or second or higher order logic (analytically definable); [12]

At this stage we have gone as far as we can go with a constructivist understanding of number. We have a number system that includes every known mathematical constant, all named transcendental numbers[13], and indeed even every *potentially* definable number. But while significantly expanded, our number set is still *countably* infinite, i.e. can be placed in one-to-one correspondence with the whole numbers.[14] Countable infinity—the cardinality of the rationals and the only form of infinity that the constructivist approach allows—turns out to be a limiting condition to the birth of the continuum. We will be forced to conclude that there is no way to obtain the continuum without triggering the admittance of the full vastness of the so-called 'uncountable infinity'.[15]

Analytically, we want a suitable model for the geometric continuum, i.e. we want a *guarantee* that our number system does not have any holes or gaps. Connectedness is the requirement that there is a continuous path between every pair of points (a, b) in a set S, such that the unit interval $[0, 1]$ can be mapped continuously into S. In symbols: $\exists\ f(t) :\ [0, 1] \to S$, s.t. $f(0) = a, f(1) = b$.[16]

It turns out that obtaining an analytically complete concept of number forces upon us a new set which is enormous beyond imagining. So long as we do not allow a somehow larger notion of infinity, there are simply not enough points with which to create a continuum, despite the expansions listed above to the rationals (FAQ5). This will mark the first encounter with a higher order of infinity, the *uncountable* infinity (FAQ5).

Our route to constructing the continuum (following Cantor) is to explicitly define, as a distinct 'number, every convergent infinite sequence of rationals whose limit is distinct (alternatively Dedekind cut), then to define the arithmetic combination of these, and finally to show that their totality is arithmetically closed and therefore by definition analytically *complete*.[17] The resulting set of 'numbers' (actually convergent sequences) is denoted by $\mathbb{R}$ and called the *'real'* numbers, as it is unique under isomorphism. With this is constructed a precise mathematical model for both space (the continuum) and time (infinitely divisible durations).[18]

Whereas the first shock was that the rationals are insufficient for geometry, there is now a second shock: Cantor's diagonal argument shows that $\mathbb{R}$ cannot be put into one-to-one correspondence with the rationals, meaning that it is of a so-called *uncountable* infinity, i.e. a higher order of infinity than the countable infinity. This is because $\mathbb{R}$ was built to include the set of all possible decimal expansions in between every unit interval, i.e. all possible infinite sequences of digits. and so contains essentially the powerset of the naturals. We know that the powerset of a set, whether finite or infinite, can never be put into one-to-one correspondence with its generating set, i.e. in symbols, $|\wp(S)| \neq |S| \; \forall S$. We are thus forced to accept the disconcerting fact that by filling in all possible gaps to ensure the continuum (spatially continuous model), we have introduced a vast, uncountable infinity of numbers which can neither be computed nor even defined.

An example of the uncountably infinite new numbers that have been added into $\mathbb{R}$, but which cannot be constructed, computed, or even defined, is Chaitin's constant, whose binary encoding is based on deciding a sequence of halting problems. Since the halting problem is itself undecidable, the number is therefore also undefinable.[19]

This vastness is a source of many paradoxes ('monsters') in analysis and topology.[20] As an example, the open interval $(-\epsilon, +\epsilon) \; \forall \epsilon > 0$ can be shown to have the same cardinality as the entirity of $\mathbb{R}$ and furthermore the same cardinality as $\mathbb{R}^n \; \forall n \in \mathbb{N}$ (this is shown using a space- and volume- filling construction due to Peano). Cardinality, not dimension, is now the key concept that determines the true size of sets. The result is a collapse of infinite sizes into a countably infinite hierarchy, with countable infinity as the smallest infinite cardinality $\aleph_0$, and each power-set of the previous infinite cardinal gives the next, i.e. $\aleph_1 = 2^{\aleph_0}$, $\aleph_2 = 2^{\aleph_1}, \ldots$. An unresolved question is whether there exists a set and cardinality greater than countable $\aleph_0$ but smaller than the cardinality of the continuum $\mathfrak{c} = 2^{\aleph_0} = \aleph_1$, i.e. whether the next largest infinite cardinal is that of the continuum. This is the Continuum Hypothesis, shown to be independent of the current standard model of set theory axiomatized by the ZFC axioms. One view among logicians is that settling CH one way or the other will require additional axioms for set theory—but what their justification might be is not yet clear.[21]

To bring algebraic closure to $\mathbb{R}$, we must add the complex field extension with the usual definition of addition of like symbols, and the binomial definition of multiplication described earlier. This forms the set of complex numbers $\mathbb{C} = \mathbb{R}[i := \sqrt{-1}]$, also definable as a pair (a, b) of real numbers in a vector space with basis $\{1, i\}$. This is the smallest analytically complete *and* algebraically closed field. All complex algebraic operations are now permitted, including taking roots of negative real numbers, and powers and roots of rationals, reals, and even complex numbers themselves. Importantly, every polynomial now splits into linear factors in $\mathbb{C}$ or, in other words, has all its roots (the Fundamental Theorem of Algebra). But algebraic closure for $\mathbb{C}$ comes at the cost of losing well-ordering and increasing the occurrence of branch cuts.[22]

$\mathbb{C}$ has another remarkable property when viewed geometrically: multiplying by a complex number z is equivalent to inducing motion: a stretching when z is pure real, rotation when pure imaginary, and both together when z is general.[23] This leads to a direct analogy between complex numbers and two-by-two matrices:

$$z = (x + iy) = re^{i\theta} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Inspired by this success with $\mathbb{C}$, we discover the quaternions $\mathbb{H}$ as a pair of complex numbers to be an algebraically closed field of dimension four, interpretable as stretching and rotation in three-dimensional space, and reinforcing the link between number and motion. The geometric link means $\mathbb{H}$ can also be modelled using 3×3 matrices. But this is gained at a cost of losing commutativity.

Going further still, we discover the octonions $\mathbb{O}$ of dimension eight as a pair of quaternions, where now associativity is sacrificed in their multiplication.[24] Remarkably, this construction in powers of 2 stops here — $\mathbb{O}$ are the highest dimension possible normed division ring. Equally remarkably, octonions have deep connections to physics string theory and geometry. Further generalizations of the number concept are vectors, matrices, tensors, and abstract rings[25] and groups. (FAQ10)[26]

We have now seen that the concept of number embodied in $\mathbb{R}$ and $\mathbb{C}$ describes space via the continuum model (continuous coordinates using tuples of numbers forming multi-dimensional (vector) spaces), describes time via infinitely divisible real quantities, and describes motion using $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, featuring rotation and stretching in 2-, 3- and 8-dimensional space.[27]

Notes

[1]The Greeks shied away from number after discovering the incommensurability of a unit and the diagonal of a unit square, separating in their science space and length from whole number. Motion was geometric, through space. Fermat and Descartes returned number to space through the notion of coordinates that vary continuously. Cantor, Cauchy, Weierstrass created the analytical (arithmetic) justification for the reals. Newtonian conception of time was a single flow, linear, unchanging, constant. Minkowski and Einstein unified space and time into a single physical context. The complex numbers brought in rotation, continued through the quaternions. Abstract algbra has extracted the essential properties to allow any sort of computation, and group theory has captured motions in this. Through algebraic structures, the arithmetization of the reals, analytic geometry, and the topological concepts of continuity, connectedness, and completeness, number has now unified both motion and space.

[2]These are not required to have inverses, e.g. rock-paper-scissors, though in general, we move naturally to find systems in which every operation can be reversed, since we believe there should always be a way back – if we go from A to B, there should also be a way back to A. We prefer unique inverses, but we know that there are zero-divisors, e.g. two non-zero matrices multiplied to get zero matrix (exercise). Just as there are multiple paths to get from A to B, so there are multiple ways to get back.

[3]Rock · Scissors = Rock; Rock · paper = Paper; Scissors · Paper = Scissors. Note commutativity. But not associativity: (Rock paper) · Scissors = Paper · Scissors = Scissors $\neq$ Rock = Rock · Scissors = Rock · (Paper Scissors). `https://www.tofugu.com/japan/japanese-rock-paper-scissors/` Note–tournaments are notoriously non-associative algebras: A beats B, B beats C, but C beats A. One might even say that life in general is non-associative because of the complexity of how different talents interact. So associativity holds when A,B,C are simple, single-attribute objects, or when there is no element of chance. Voting is another example. The cycles resemble warfare: cavalry effective against archers, archers against pikemen, and pikemen against cavalry. There are connections with game theory, non-linear dynamics, graph theory (for extensions with more elements), parity algorithms, and automated game playing algorithms which do historical analysis). What makes a system non-associative? Why are the Octonions non-associative when the Quaternions are associative? `https://en.wikipedia.org/wiki/Rock-paper-scissors`

[4]An irrational number must have an infinite and non-repeating decimal expansion. A rational number has either a finite expansion, or if infinite but has a finite block that repeats indefinitely.

[5]FAQ9. It is worth noting that computational algebra systems (CAS) are able to work entirely in rationals. How do they represent the entire set, both the very large and very small? How are they represented under-the-hood in finite precision (e.g. 64-bit)?

[6]FAQ: floating point numbers and their error analysis. Describe for an 8-bit, 16-bit, and 64-bit computer.

[7]FAQ1: Infinite sets are problematic, even countably infinite ones such as the whole numbers, as they require second order logic for their construction (first order logic is only capable of working with finite sets). And while a finite set with only addition can escape Godels' incompleteness theorem, infinite set and two operator arithmetic implies a logic structure that is incomplete. FAQ2: development and improvement of these numerical methods.

[8]These are irrational because the elliptic integrals[?] involve roots, which we know are irrational.

[9]Indeed, this means this simple equation is insoluble for most composite numbers. It's actually rather unusual to have a perfect square. While there are infinitely many of these, they are ever more sparsely distributed in $\mathbb{N}$.

[10]FAQ: Closure is an algebraic property. The rationals are a field. It will turn out that the largest reasonable extensions are at least a division algebra, of which there are only 4 of finite dimension, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. See Division Algebra: `https://en.wikipedia.org/wiki/Division_algebra`, Algebra over a field, `https://en.wikipedia.org/wiki/Algebra_over_a_field`, Algebraically closed field, `https://en.wikipedia.org/wiki/Algebraically_closed_field`

[11]majority of real numbers are non-computable in the sense, e.g. of Specker sequences, which don't have a computable supremum despite being bounded and strictly increasing

[12]Definable number: `https://en.wikipedia.org/wiki/Definable_real_number`. Arithmetic closure is automatic by definition. Are there any non-definable numbers?

[13]Since to know such a number is to be able to specify it with a definition

[14]Mathematical constants: `https://en.wikipedia.org/wiki/Mathematical_constant`

[15]This is because by constructive methods we could only reach countable infinity (recall even the computable and definable numbers obtained above were countably infinite).

[16]Connectedness as a concept relies on continuity. Does connectedness require the axiom that the unit closed $[0,1]$ interval is connected? Does continuity require the reals—i.e. can it be defined on the rationals? What about functions defined solely on the rationals, and irrationals, e.g. $f(x) = 0$ if rational and 1 if irrational. Such a function is clearly not continuous everywhere, but is it continuous anywhere? This has to do with the density of the rationals and irrationals in the reals. Why is a discrete set disconnected?

[17]Is it acceptable from a set theory perspective to consider ll' such sequences? Is this correct - $\mathbb{R}$ are complete by definition? Revisit Rudin.

[18]It is a connected set in the sense of having a mathematically continuous path between every pair of points. The definition of continuity is itself based on the properties of the rationals and the axiom that the interval (0,1) is connected ensures that there is no hole, no minimum sized gap in the set of numbers. Implicitly we are using a distance measure, a metric, relating distance between points with coordinates, numbers.

[19]Looking at Chaitin's constant, one could make an argument for undefinable numbers being analogous to decision problems based on free will - e.g. binary encoding what a robot would do at all decision points in an infinite maze (or for a person, all decision points they face in life), and potentially all of the knock-on possibilities (considering all branches).

[20]See Lakatov for discussion of monsters in mathematics. See counterexamples in analysis and topology for a catalog of these. See remarks.

[21]See Solomon Feferman

[22]which takes precedence, $1+i$, or $1-i$? Working in the reals saw branch cuts occur when taking the inverse of a one-to-many function such as x^2 or $\sin(x)$, and by convention a primary or canonical branch. In $\mathbb{C}$, branching occurs much more frequently, as all elementary functions (logarithmic, exponential, trigonometric, and hyperbolic) are one-to-many. (In $\mathbb{R}$, log and exp were one-to-one.).

[23]The Argand diagram shows complex numbers as vectors. The Euler identity and its derivation explains why multiplication in $\mathbb{C}$ is expansion and rotation:

$$z = x + iy = r\text{cis}(\theta) = \cos(\theta) + i\sin(\theta) = re^{i\theta}$$

by recognizing x, y as projections in 2-D argand space, then replacing cos and sin with their infinite series expansions and comparing with the series expansion of exp formally evaluated at i (FAQ6)/

[24]See John Baez.

[25]It's actually quite remarkable and a significant constraint to require a set with two operations, an identity for both, inverses for both, and the fact that there are no zero divisors. Why do the sedenions (the next algebra above octonions) lose divisibility?

[26]There are two essential elements in the number concept: 1) the possession of an algebraic structure, and 2) the correspondence between numbers, points, and lengths. Generalizations then occur algebraically for objects with closed binary operations (e.g. a group), and geometrically/analytically for points having a geometric structure (e.g. vectors, Lie algebras).

[27]The view of complex numbers as inducing motions is the perspective of conformal mapping in complex variables theory.

Recommended Reading.

(1) [20] covers much of the material in this article in greater depth by a first rate research mathematician and prize winning expository writer of mathematics.
(2) [9] covers the foundations of our number systems in exceptional clarity by one of the foremost logicians and authorities on the subject.
(3) [16] covers the construction of the number systems in an ultra-dry, Bourbaki style.
(4) [4] is an outstanding expository article on the octonions and their connections to higher algebra and geometry.
(5) [19] is another exposition that covers the quaternions and octonions and their colorful history, written by a master expositor.
(6) [14]
(7) [4] published in Scientific American, an interview with Baez

References

[1] Archimedes. The sand reckoner. *Online various places, and In "The World of Mathematics" by James R. Newman (pp420–430).* `http://www.math.uwaterloo.ca/navigation/ideas/reckoner.shtml`, also `http://www.sacred-texts.com/cla/archim/sand/sandreck.htm`.

[2] John Baez and John Huerta. The strangest numbers in string theory. *Scientific American*. A remarkably clear exposition of supersymmetry theories in physics and how the octonions fit in. `http://math.ucr.edu/~huerta/`.

[3] John Baez and John Huerta. Algebra of grand unified theories. *Bulletin of the American Mathematical Society*, 47:483–552, 2010. A clear exposition of the mathematics behind grand unified theories in physics, using Lie Theory, and explaining the interaction of the fundamental particles (matter and force) in the universe according to the Standard Model of physics, which has recently received a tremendous boost by the detection of the Higgs boson in the Large Hadron Collider. `www.ams.org/journals/bull/2010-47-03/S0273-0979-10-01294-2/`.

[4] John C. Baez. The octonions. *Bulletin of the American Mathematical Society*, 39:145–202, May 2001. Published in Bull, Amer.Math.Soc, electronically in Dec 21, 2001, in the journal in 2002, Accepted into ArXiv on 23 Apr 2002. Available from the author's website: `http://math.ucr.edu/home/baez/octonions/` or from the journal at `http://www.ams.org/journals/bull/2002-39-02/S0273-0979-01-00934-X/home.html`,.

[5] Richard Courant. What is mathematics? (an article). 1941. This is the first five pages of the book of the same name [**?**]. Essay is available at `http://www.math.uga.edu/~azoff/ffds.pdf` Article includes an explanation of how the viewpoint of modern mathematics is the same as that of modern science, and how this has necessarily led to the axiomatic / postulational formulation of mathematics.

[6] Richard Dedekind. *Essays on the Theory of Numbers*. Open Court Publishing Company, 1888; 1901; 1963. Contains two essays: "What are numbers and what should they be?", and an essay on transfinite numbers.

[7] Freeman Dyson. Missed opportunities. *Bulletin of the American Mathematical Society*, 78(5), 1972.

[8] Solomon Feferman. The development of programs for the foundations of mathematics in the first third of the 20th century. *Available from the Internet at the author's website*. This is one of the BEST accounts of the various programmes in mathematics explained by a logician who cares about clarity and not just dogma.

[9] Solomon Feferman. *The Number System: Foundations of Algebra and Analysis*. AMS Chelsea, 1964, 1989.

[10] Solomon Feferman. Does mathematics need new axioms? *American Mathematical Monthly, Vol. 106, No. 2*, pages 99–111, Feb 1999. Text of an invited AMS MAA lecture, Joint Annual Meeting, Jan 10, 1997. Available as PDF from author's website: `http://math.stanford.edu/~feferman/papers/newaxioms.pdf`.

[11] A.O. Gelfond. *Transcendental and Algebraic Numbers*. Dover, translated to english by leo f. boron edition, 1958 (First Russian Edition); 1960 (Dover publication). This is one place where the full (advanced working mathematician) proofs are given for various transcendency results: a. that general algebraic numbers are poorly approximated by algebraic

numbers from a fixed algebraic field; b. that e is a transcendental number; c. that a variety of other functions and number forms are transcendental.

[12] Jekuthial Ginsburg. New light on our numerals. *Bulletin of the American Mathematical Society; Vol.23; No.8;*, pages 366–369, May 1917. Available as PDF from http://projecteuclid.org/euclid.bams/1183424084.

[13] Jan Gullberg. *Mathematics from the Birth of Number: The history of mathematics with a broad treatment of its foundations*. W.W.Norton, 1997. This is a beautifully crafted, intelligent, engaging and fascinating romp through mathematics and mathematical culture. It presents in an accessible manner, in one place, a well-structured and tantalizing journey of progress through mathematical history. Highly recommended. Recommend [15] as a more detailed second reference on the history of numbers.

[14] John Huerta. Misc. papers, website. His webpage: http://math.ucr.edu/~huerta/.

[15] George Ifrah. *The Universal History of Numbers: From pre-history to the invention of the computer*. Harvill, translated from the french by david bellos edition, 1994; 1998. This is an interesting anthropological study of the number concept, its phenomenology, origins, and historical evolution throughout world cultures. This is a much more detailed study of material covered quite admirably in [13]. Recommend [13] as the first reference, this if more detail is needed.

[16] Edmund Landau. *Foundations of Analysis: The Arithmetic of Whole, Rational, Irrational, and Complex Numbers, A Supplement to Text-Books on the Differential and Integral Calculus*. Chelsea Publishing Company, 4th, translated to english by f. steinhardt edition, 1930 (1st), 1951 (4th).

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[18] John Stillwell. *Mathematics and Its History*. 3rd edition, 1989 (1st ed), 2001 (2nd ed), 2010 (3rd ed).

[19] John Stillwell. The story of the 120-cell. *Notices of the AMS*, pages 17–24, January 2001.

[20] John Stillwell. *Yearning for the Impossible: The Surprising Truths of Mathematics*. AK Peters, 2006. This covers much of the material in my article: The evolution of Numbers, in much more detail.

[21] John Stillwell. *The Real Numbers: An Introduction to Set Theory and Analysis*. Springer, Undergraduate Texts in Mathematics, 2013.

The evolution of mathematics might be seen as an ever-increasing series of abstractions, or alternatively an expansion of subject matter. The first abstraction was probably that of numbers. The realization that two apples and two oranges have something in common was a breakthrough in human thought. In addition to recognizing how to count physical objects, prehistoric peoples also recognized how to count abstract quantities, like time â€” days, seasons, years. Arithmetic (addition, subtraction, multiplication and division), naturally followed. Monolithic monuments testify to knowledge of geometry