ORIGINS OF MATHEMATICAL THINKING: A SYNTHESIS

Uri Leron

Technion – Israel Institute of Technology

<uril@tx.technion.ac.il>

Abstract: *This paper synthesizes recent research from several disciplines, concerning the cognitive, social and biological origins of mathematical thinking. In order to put the emerging views into a coherent framework, I distinguish three levels of mathematics, each with its own thinking mechanism. First, the ability to do* ***Rudimentary Arithmetic*** *is hard-wired in the brain and is processed by a "number sense", just as colors are processed by a "color sense". Second,* ***Informal Mathematics*** *is processed by the same mechanisms that make up our everyday cognition, such as imagery, natural language, thought experiment, social cognition and metaphor. Third, there is some evidence that thinking at the level of* ***Formal Mathematics*** *may actually conflict with some of our mind's "natural" cognitive mechanisms.*

Introduction

In this paper I consider the following (admittedly vague) question:

> ***Is mathematical thinking a natural extension of common sense, or is it an altogether different kind of thinking?***

The possible answers to this question are of great interest and importance for both theoretical and practical reasons. Theoretically, this is an important special case of the general question of how our mind works. In practice, the answers to this question clearly have important educational implications.

Recently, several books and research papers have appeared, which bear on this question, so that the possible answers, though still far from being conclusive, are less of a pure conjecture than they had previously been. These new studies have inquired into the cognitive and biological origins of mathematical thinking and have come from research disciplines as varied as neuroscience, cognitive science, cognitive psychology, evolutionary psychology, anthropology, linguistics and ethology; their subjects were normal adults, infants, animals, and patients with brain damage. This body of research is not well-known in the mathematics education community despite its relevance and importance, hence one (secondary) goal of this paper is to give a brief overview of its main methods and results.

The conclusions of the various researchers seem at first almost contradictory: Aspects of mathematical cognition are described as anything from being embodied to being based on general cognitive mechanisms to clashing head-on with what our mind has been "designed" to do by natural selection over millions of years.

However, these seeming contradictions all but fade away once we realize that "mathematics" (and with it "mathematical cognition") may mean different things to different people, sometimes even to the same person on different occasions. In fact, the main goal of this paper is to show that all this multifaceted research by different people coming from different disciplines, may be neatly organized into a coherent scheme once we exercise a bit more care with our distinctions and terminology.

To this end, I will distinguish three levels of mathematics, called here ***rudimentary arithmetic***, ***informal mathematics*** and ***formal mathematics***, each with its own different thinking mechanisms[1]. When interpreted within this framework, the research results show that while certain elements of mathematical thinking are innate and others are easily learned, certain more advanced (and, significantly, historically recent) aspects of mathematics—formal language, de-contextualization, abstraction and proof—may be in direct conflict with our mind's "natural" thinking.

Because of space limitations I cannot even begin to do justice to the many facets and subtleties of this ingenious research. I can only give here a brief (and much over-simplified) overview of this vast and rich area, also omitting references to the primary sources. For a fuller account, the reader is referred to the excellent expositions—and full references—in Dehaene (1997) and Butterworth (1999) for Level 1; Lakoff & Núñez (2000) and Devlin (2000) for Levels 1 and 2; Cosmides & Tooby (1992, 1997) for Level 3.

Level 1: Rudimentary arithmetic

Rudimentary arithmetic consists of the simple operations of subitizing, estimating, comparing, adding and subtracting, performed on very small collections of concrete objects. Research on infants and on animals, as well as brain research, indicates that some ability to do mathematics at this level is hard-wired in the brain and is processed by a 'number sense', just as colors are processed by a 'color sense'. Excellent syntheses of this research are Dehaene (1997) and Butterworth (1999).

It is hard to prove that some feature is an "adaptation", brought about by evolution via natural selection, but a strong case can be made by showing that three conditions are fulfilled: One, the feature in question could have conferred a clear survival advantage on our stone-age hunter-gatherer ancestors; two, some version of this feature exists in our non-human relatives; three, babies already exhibit this feature even before they had a chance to learn it from their physical or social environment.

Indeed, it is easy to imagine how rudimentary arithmetic could help survival for our ancestors: in keeping count of possessions and in estimating amount of food (going for the tree with more fruit) and number of enemies.

[1] Strictly speaking, 'formal' and 'informal' ought to refer not to the mathematical subject matter itself but to its presentation and re-presentation. In many cases these may in fact describe two facets of the *same* piece of mathematics.

There are many experiments showing that some animals (such as chimpanzees, rats and pigeons) have 'number sense'. A striking example is an experiment by Karen McComb and her colleagues (cf. Butterworth, 1999, pp 141-2) showing that when a female lion at the Serengeti National Park in Tanzania detects the roar of unfamiliar lions invading her territory, she will decide to attack only if the number of her sisters nearby on the territory is greater than the number of invaders. This is all the more remarkable because she seems to compare the two numbers across sense modalities: she *hears* the intruders but *sees* (or memorizes) her sisters. "Thus she has to abstract the numerosity of the two collections—intruders and defenders—away form the sense in which they were experienced and then compare these abstracted numerosities." (ibid)

It seems at first all but impossible to establish what mathematical facts a very young baby knows, but developmental psychologists using ingenious research methods have nonetheless managed to establish a body of firm results. See Dehaene (1997) for a comprehensive survey and reference to the original research literature. The following brief summary is taken (with some omissions) from Lakoff and Núñez (2000, pp 15-16).

1. At three or four days, a baby can discriminate between a collection of two and three items. […]
2. By four and half months, a baby "can tell" that one plus one is two and that two minus one is one. […]
3. These abilities are not restricted to visual arrays. Babies can also discriminate numbers of sounds. At three or four days, a baby can discriminate between sounds of two or three syllables. […]
4. And at about seven months, babies can recognize the numerical equivalence between arrays of objects and drumbeats of the same number. […]

There are too many details and variations to do justice to this intricate research here, but the reader can get some idea from a brief description of one of the main methods used: timing the baby's gaze and the ***violation-of-expectation*** research paradigm. When a baby looks for a while at a repeating or highly expected scene, it will get bored and will look at the scene for shorter and shorter periods (a phenomenon called ***habituation***). When the scene suddenly changes, or something unexpected happens, the baby's gaze duration (called ***fixation time***) will become measurably longer. Researchers moved behind a screen, in front of the baby's eyes, one puppet and then another, and then lifted the screen to reveal what's behind it. Babies typically looked significantly longer (i.e., were surprised) when they saw one puppet (or three) behind the screen, as compared to two. This experiment was repeated with many variations, with the inevitable conclusion that, in a sense, babies are born with the innate knowledge that one and one makes two.

Level 2: Informal Mathematics

This is the kind of mathematics, familiar to every experienced teacher of advanced mathematics, which is presented to students in situations when mathematics in its most formal and rigorous form would be inappropriate. It may include topics from all mathematical areas and all age levels, but will consist mainly of "thought experiments" (Cf. Lakatos, 1978; Tall, 2001; Reiner & Leron, 2001], carried out with the help of figures, diagrams, analogies from everyday life, "typical" examples, and students' previous experience. For example, when teaching group theory, many instructors preface the formal presentation of the proposition $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$ by the following intuitive analogy: Suppose you put on your socks and then your shoes. If you now want to undo this operation, you need to first take off your shoes and then your socks. Thus to find the inverse of a combined operation you need to combine the individual inverses ***in reverse order***.

Some recent research, as well as classroom experience, indicate that informal mathematics ***is*** an extension of common sense, and is in fact being processed by the same mechanisms that make up our everyday cognition, such as imagery, natural language, thought experiment, social cognition and metaphor. That mathematical thinking has "hijacked" older and more general cognitive mechanisms is in fact only to be expected, taking into account that mathematics (except for rudimentary arithmetic) has been around for only about 2500 years – a mere eye blink in evolutionary terms. (Our brains have evolved over millions of years, and are believed by experts to have been essentially fixed in their current form for at least 50,000 years.)

Two recent books—by Lakoff & Núñez (2000) and by Devlin (2000)—present elaborate theories to show how our ability to do mathematics is based on other (more basic and more ancient) mechanisms of human cognition. Significantly for the thesis presented here, both theories mainly seek to explain the thinking processes involved in Level 2 mathematics, so that their conclusions need not apply to Level 3. In fact, as I explain in the next section, there are reasons to believe that their conclusions (as far as the general population is concerned) do not apply to Level 3 mathematical thinking[2].

Lakoff and his colleagues have for many years argued convincingly the case for metaphor as a central mechanism in human cognition. Recently, Lakoff & Núñez (2000) have extended this argument to a detailed account on how mathematical cognition is first rooted in our body via embodied metaphors, then extended to more

[2] The authors are not always explicit on the scope of mathematics they discuss, but see e.g., "I am not talking about becoming a great mathematician or venturing into the heady heights of advanced mathematics. I am speaking solely about being able to cope with the mathematics found in most high school curricula." (Devlin, 2000, p. 271); and "Our enterprise here is to study everyday mathematical understanding of this automatic unconscious sort […]" (Lakoff & Núñez, 2000, p. 28).

abstract realms via "conceptual metaphors", i.e., inference-preserving mappings between a source domain and a target domain, where the former is presumably more concrete and better-known than the latter. In their account they thus show (more convincingly in some places than in others) how mathematical cognition builds on the same mechanisms of our general linguistic and cognitive system.

According to this theory, our conceptual system is mostly built "from the bottom up", starting from our embodied knowledge and gradually building up to ever more abstract concepts. However, an interesting twist to this picture has been suggested by Tall (2001). Since many parts of modern mathematics (especially those dealing with the various facets of infinity) go strongly against our "natural" intuitions, it is hard to build appropriate understandings of them solely via metaphorical extensions of the learner's existing cognitive structures. (The research literature abounds with examples of students' "misconceptions" arising from such clashes between natural intuitions and the formal theory.) As Tall shows, we need to also take into account a process going in the opposite direction. Some of the results of the formal axiomatic theory (called "structure theorems") may feed back to develop more refined intuitions (or embodiments) of the concepts involved.

Devlin (2000) gives a different account than Lakoff & Núñez, but again one attempting to show how mathematical thinking has "hijacked" existing cognitive mechanisms. His claim is that the metaphorical "math gene"—our innate ability to learn and to do mathematics—comes from the same source as our linguistic ability, namely our ability for "off-line" thinking (basically, performing thought experiments, whose outcome will often be valid in the external world). Devlin in addition gives a detailed evolutionary account of how all these abilities might have evolved[3]. I find Devlin's account rather convincing, provided you limit it to informal mathematics. In other words, his account fits well situations in which people do mathematics by constructing mental structures and then navigate within those structures[4], but not situations where such structures are not available to the learner. For example, it is hard to imagine any "concrete" structure that will form an honest model of a uniformly continuous function or a compact topological space.

Level 3: Formal Mathematics

The term "formal mathematics" refers here not to the contents but to the form of advanced mathematical presentations, with their full apparatus of abstraction, formal language, de-contextualization, rigor and deduction. The fact that understanding formal mathematics is hard for most students is well-known, but my question goes farther: is it an extension (no matter how elaborate) of common sense or an altogether different kind of thinking? Some research, as well as the persistent failure of many bright college students to master it, suggest that the thinking involved in formal mathematics is ***not*** an extension of common sense; that it may in fact sometime clash

[3] Devlin is relying here substantially on Bickerton's (1995) account of the evolution of language.
[4] See in this connection his "mathematical house" metaphor on p. 125.

head-on with human "natural" thinking. Looking through the lens of the young and exciting (though still controversial) discipline of evolutionary psychology, this evidence suggests that some parts of modern mathematical thinking may clash with what our mind/brain has been designed by natural selection to do "naturally"[5].

Cosmides and Tooby (1992, 1997) have used the Wason card selection task (which tests people's understanding of "if P then Q" statements; cf. Wason, 1966; Wason & Johnson-Laird, 1972) to uncover what they refer to as people's evolved reasoning "modules". In a typical example of the card selection task subjects are shown a row of four cards, say [A] [T] [6] [3], and are told that each card has a letter on one side and a number on the other. The subjects are then presented with the rule, "if a card has a vowel on one side, then it has an even number on the other side", and are asked the following question: ***What card(s) do you definitely need to turn over to see if any of them violate this rule?*** The infamous result is that over 75% of the subjects, including college students in scientific disciplines, gave an incorrect answer. The percentage depends somewhat on the content of P and Q and on the background story. (The correct answer is: [A] and [3] .)

Cosmides and Tooby have presented their subjects with many versions of the task, all having the same logical form "if P then Q", but varying widely in the contents of P and Q and in the background story. While the classical results of the Wason Task show that most people perform very poorly on it, Cosmides and Tooby have found that their subjects performed rather successfully on tasks involving conditions of "social exchange". In social exchange situations the individual receives some benefit and is expected to pay some cost. In the Wason experiment they are represented by statements of the form "if you get the benefit, then you pay the cost" (e.g., if I give you 20$, then you give me your watch). A cheater is someone who takes the benefit but do not pay the cost. Cosmides and Tooby explain that when the Wason task concerns social exchange, a correct answer amounts to detecting a cheater. Since subjects performed correctly and effortlessly in such situations, Cosmides and Tooby have theorized that our mind contains evolved "cheater detection algorithms".

Most strikingly for mathematics education, Cosmides and Tooby have also tested their subjects on the so-called "switched social contract" (mathematically, the converse "if Q then P"), in which the correct answer by the logic of social exchange is different from that of mathematical logic (cf. Cosmides and Tooby, 1992,

[5] Evolutionary psychology, unlike sociobiology and behavior genetics, investigates the evolutionary roots of psychological attributes that are shared by all humans, regardless of their particular culture, education, gender, geographical location or race. In this sense it may be said to study universal human nature. The assumption is that understanding the product of complex design may be greatly aided by the consideration of what problem that product was designed to solve. This is not to deny that much of our psychology is shaped by our cultural and social environment. There are many subtle and emotionally laden issues here, and many misconceptions exist about the nature of this enterprise, which cannot be addressed here. See the clear and accessible introduction in Cosmides & Tooby (1997). For a more comprehensive exposition, see Pinker (1997, 2002) and Plotkin (1998).

especially pp. 187-193). The results were that their subjects overwhelmingly chose the former, not the latter. It seems that when conflict arises, the logic of social exchange overrides mathematical logic. This adds a new level of support, prediction and explanation to the many findings (e.g. Hazzan & Leron, 1996) that students are prone to confusing between mathematical propositions and their converse.

Conclusion

So, ***is mathematical thinking an extension of common sense***?

We can now summarize the answer a little more precisely. According to contemporary thinking in cognitive science (e.g. Lakoff & Núñez, 2000; Pinker, 1997,2002), common sense is what our mind does "naturally". It is a set of procedures—such as learning mother tongue, recognizing faces, negotiating everyday physical and social situations, and using rudimentary arithmetic—that have evolved by natural selection because they had conferred some survival and reproductive advantage on our stone-age hunter-gatherer ancestors.

These procedures are natural in the sense that they are either innate or are easily and spontaneously learned by all human beings with normal development, regardless of geography, culture, education, race or gender. Because modern mathematics—like other artifacts of modern civilization such as writing or driving—is too young in evolutionary terms, it is clear that we don't have cognitive mechanisms that evolved specifically for mathematical thinking. To the extent that we at all can do mathematics, it must be based on older mechanisms that have been hijacked by our mind for this new purpose.

The research surveyed in this paper shows that this is indeed the case in what I have called Informal Mathematics: It is processed by the common sense mechanisms of language, social cognition, mental imagery, thought experiment and metaphor. Classroom experience, too, indicates that students have little trouble making sense of mathematics as long as it is presented through familiar examples and analogies. The same classroom experience, however, indicates that students do have a lot of trouble with the switch to Formal Mathematics. It seems as though our mind contains no cognitive mechanism that could be hijacked for this purpose. This doesn't mean it can't be done: after all, people do achieve such unnatural feats as juggling 10 balls while riding on a bicycle or playing a Beethoven piano sonata. It does mean that the huge amount of effort and practice needed to get there requires an equally huge amount of ***motivation*** from the learner, and therein lies the trouble.

The research from evolutionary psychology (Cosmides & Tooby, 1992) hints that the situation with Formal Mathematics may be even worse that that. Not only do we not have cognitive modules that can be marshaled for this kind of thinking; it may even be in direct clash with thinking we do find natural, such as negotiating social situations.

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