Borrelli goes on to discuss what she describes as philosophical aspects of the astrolabe in Chapter 5. Her point here is that the astrolabe served not so much as a monastic tool; after all, hardly any of the monks' duties would have called for one. Rather, by creating a device that mimics motions in the natural world, astrolabe practitioners were participating in a sort of architectural or mechanical rationality (hence the subtitle of the book) that provided a bridge between geometry and reality. Hence the appearance in the texts of a couple of ridiculously impractical suggested uses of the astrolabe, including timings of events only a few minutes in duration: the goal of the passage is not actually to have readers perform the procedure, but to make a mathematical connection between the heavens and the earth. The monks' interaction with the *machina mundi* took place in contact with the physical device itself, and the texts were imperfect reflections of this interaction.

Whether or not one accepts Borrelli's specific contentions regarding the nature of the manuscripts, diagrams and astrolabes that she examines, one must recognize that she raises a good point. Taking the texts as the sole representation of what the monks were thinking and doing is liable both to mislead us, and to cause us to misunderstand the very texts that we advocate.

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The St. Petersburg School of Number Theory

By B.N. Delone. Translated from the Russian by Robert Burns. History of Mathematics, vol. 26. Providence/London (American Mathematical Society/London Mathematical Society). 2005. ISBN: 978-0-8218-3457-6. xv + 278 pp. US$62.00

When I mentioned to a colleague that I was writing a review of a book entitled *The St. Petersburg School of Number Theory*, his first reaction was to ask, "Is it a book about Euler?" This is certainly a natural response, given the association of Leonhard Euler with St. Petersburg through his long tenure at the St. Petersburg Academy of Sciences.

Although Euler does make a few guest appearances, the purpose of this volume is to discuss the groundbreaking research in number theory conducted by Chebyshev, Korkin, Zolotarev, Markov, Voronoĭ, and Vinogradov at St. Petersburg University in the 19th and 20th centuries. In fact, the book is a centennial volume, since Chebyshev joined the faculty of the University in 1847 and the original Russian version of this volume was published in 1947. All of the above named mathematicians (including the author Boris Delone, himself an important number theorist) studied and later taught at St. Petersburg University.

This volume (which was ably translated from the Russian by Robert Burns and contains a foreword by Michael Rosen) is intended to summarize the work done in number theory by this illustrious group of mathematicians. This is not a sourcebook. Specific works of the mathematicians are outlined in some detail, followed by commentaries, usually by Delone or in some cases by his colleague B.A. Venkov. Quadratic forms are a primary theme connecting the works of several of these mathematicians, but Delone assumes little specific knowledge of number theory on the part of the reader. In fact, a highly motivated undergraduate would be able to read most of this book.

In this volume one reads about the history of determining the minima of positive definite and indefinite quadratic forms, the classification of such forms, and the foundations of Minkowski's geometry of numbers. All of this is covered plus some very interesting analytic number theory by Chebyshev and Vinogradov.

The first mathematician discussed is Panuftiĭ L'vovich Chebyshev (1821–1894). The motivation for Chebyshev's interest in number theory has an Euler connection. In the 1840s, the St. Petersburg Academy of Sciences published a series of three volumes entitled *Commentationes arithmeticae collectae*, collecting various works of Euler on number theory. Viktor Bunyakovskiĭ enlisted Chebyshev to work as an editor on this project, after which Chebyshev began his own number-theoretic research.

Delone discusses two works of Chebyshev on prime numbers, "On the number of primes not exceeding a given number" (written in 1849 as an appendix to his doctoral dissertation) and "Mémoires sur les nombres premiers" from 1852. Legendre had conjectured that the number of primes less than x, denoted $\phi(x)$ by Chebyshev, can be approximated by

$$\frac{x}{\ln x - 1.08366}.$$

Chebyshev proved that this conjecture is false, and instead showed that for any positive real number α and any n,

$$\int_2^x \frac{dt}{\ln t} - \frac{\alpha x}{\ln^n x} < \phi(x) < \int_2^x \frac{dt}{\ln t} + \frac{\alpha x}{\ln^n x},$$

for infinitely many values of x. This result played an important role in the history of the prime number theorem in the 19th century, which culminated in the 1896 proof of the theorem by Hadamard and de la Vallée Poussin.

The theory of quadratic forms is a common theme in the research of Korkin and Zolotarev, Markov, and Voronoĭ. An n-ary quadratic form is a homogeneous polynomial,

$$Q(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j,$$

where it is assumed that the coefficients are real and symmetric, namely $a_{ij} = a_{ji}$. A binary quadratic form has two variables. A quadratic form is positive definite if $Q(x) > 0$ for any $x \neq 0$. An indefinite form will be one that is sometimes positive and sometimes negative. The determinant $D = \det(a_{ij})$ also plays a central role in the theory.

In a letter to Jacobi of August 6, 1845, Hermite proved that every n-ary positive definite quadratic form with determinant D has a minimum μ satisfying

$$\mu \leqslant c_n \sqrt[n]{D},$$

where c_n is a constant depending only on n. Hermite showed that $\left(\frac{4}{3}\right)^{(n-1)/2}$ is a sufficient upper bound for c_n, but conjectured that $c_n = \frac{2}{\sqrt[n]{n+1}}$ is more precise.

In a series of papers written in 1872, 1873, and 1877, Aleksandr Nikolaevich Korkin (1837–1908) and Egor Ivanovich Zolotarev (1847–1878) studied minima of positive definite quadratic forms. They disproved Hermite's conjecture, instead finding that the following constants are exact:

$$c_2 = \sqrt{4/3}, \quad c_3 = \sqrt[3]{2}, \quad c_4 = \sqrt{2}, \quad c_5 = \sqrt[5]{8}.$$

Delone devotes a long section to showing how the work of Korkin and Zolotarev can be translated into the language of lattices. Given a positive definite form Q, it is possible to write the form as a sum of squares of linear forms

$$\begin{aligned} Q = {} & (\lambda_{11}x_1 + \lambda_{12}x_2 + \cdots + \lambda_{1n}x_n)^2 + (\lambda_{21}x_1 + \lambda_{22}x_2 + \cdots + \lambda_{2n}x_n)^2 \\ & + \cdots + (\lambda_{n1}x_1 + \lambda_{n2}x_2 + \cdots + \lambda_{nn}x_n)^2. \end{aligned}$$

The following vectors will then form a basis for $\mathbb{R}^n$,

$$\begin{aligned} \mathbf{e}_1 &= (\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1n}), \\ \mathbf{e}_2 &= (\lambda_{21}, \lambda_{22}, \ldots, \lambda_{2n}), \\ &\vdots \\ \mathbf{e}_n &= (\lambda_{n1}, \lambda_{n2}, \ldots, \lambda_{nn}). \end{aligned}$$

A lattice is formed from the points in $\mathbb{R}^n$ which have integral coordinates relative to this basis. Thus every positive definite quadratic form has an associated lattice. Delone seems to suggest that Korkin and Zolotarev anticipated this geometry of numbers, which is generally attributed to Minkowski, who published his *Geometrie der Zahlen* in 1896 [Minkowski, 1896]. Instead the truth would seem to be that work on quadratic forms by Gauss, Hermite, H.J.S. Smith, Markov, Korkin and Zolotarev laid the foundation for Minkowski's work.

Zolotarev, who died tragically at the age of 31 in a train accident, also did research on algebraic numbers. He generalized Kummer's theory of ideal numbers in his doctoral dissertation of 1874. Zolotarev's goal in his dissertation was to consider divisibility properties of algebraic integers, namely complex numbers which are roots of a monic polynomial with coefficients in $\mathbb{Z}$. Zolotarev showed that the decomposition of algebraic integers into what he calls irreducible ideal factors is exactly analogous to the decomposition of ordinary integers into prime factors.

His major paper on algebraic integers was written in 1876, but not published until 1880, when it appeared in *Liouville's Journal*. (For some reason, Delone gives 1885 as the date of this article.) In his dissertation, Zolotarev had excluded certain polynomials from his analysis, but in the 1880 paper he completes his generalization of Kummer's work. This theory was also developed by Dedekind in 1871, who replaced ideal numbers with the concept of an ideal. Zolotarev was influenced by the *Disquisitiones Arithmeticae* of Gauss and the 1871 work of Dedekind. On the other hand, both Dedekind and Kronecker seem to have dismissed Zolotarev's work as being incomplete.

A.A. Markov (1856–1922), better remembered for his work in probability, also did significant work on quadratic forms. In his master's dissertation of 1880, he extended the previous work of Korkin and Zolotarev to indefinite quadratic forms. Motivated by Euler, he used continued fractions to examine the minima of indefinite binary quadratic forms.

Markov noted that the set of minima of all indefinite binary quadratic forms has a sharp upper bound of $\sqrt{\frac{4}{5}D}$, which is attained by the form

$$\sqrt{\frac{4}{5}D}(x^2 - xy - y^2).$$

Excluding this and all equivalent forms, the remaining forms have a sharp upper bound for their minima of $\sqrt{\frac{1}{2}D}$, which is attained by the form

$$\sqrt{\frac{1}{2}D}(x^2 - 2xy - y^2).$$

In this manner, Markov obtained a sequence of numbers $N_1 = \frac{4}{5}, N_2 = \frac{1}{2}, N_3 = \frac{100}{221}, \ldots$, called the Markov spectrum. As $k \to \infty$, $N_k \to \frac{4}{9}$ and infinitely many indefinite binary quadratic forms have minimum $\frac{2}{3}\sqrt{D}$. Markov showed that the numbers in the Markov spectrum are directly related to solutions of the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz.$$

One of Markov's students was Georgiĭ Fedoseevich Voronoĭ (1868–1908), whose own student was the author of this book. One of Voronoĭ's areas of research was the problem of solving Pell's equation $x^2 - \Delta y^2 = 1$. Wallis, Euler and Lagrange had all worked on the problem by finding decompositions of Δ as a periodic continued fraction. Since $x^2 - \Delta y^2 = (x + \sqrt{\Delta}y)(x - \sqrt{\Delta}y)$, this is equivalent to finding units in a quadratic number field. In his doctoral dissertation of 1896, Voronoĭ generalized the continued fraction algorithm of Euler to find units of cubic fields. Delone presents this work in a geometric format, saying "We observe first of all that without doubt Voronoĭ framed and elicited his results geometrically" (p. 142). Since Voronoĭ's dissertation was published the same year as Minkowski's work on the geometry of numbers, one may be skeptical of Delone's claim.

Voronoĭ's best known works are his treatises of 1908 and 1909 on primitive parallelohedra. It is there that he defines the concept we now call the *Voronoi cell*. Given a lattice L in $\mathbb{R}^n$, let $c \in L$. Define $V(c)$ to be the set of points in $\mathbb{R}$ which are closest to c. The set $V(c)$ is called a Voronoi set. It is a convex polytope and the closure of the set $\{V(c)|c \in L\}$ equals $\mathbb{R}^n$. This concept plays an important role in the geometry of numbers, since it gives a way to classify lattices and their associated quadratic forms on the basis of their Voronoi cells. See Conway's book for a contemporary treatment [Conway, 1987].

The work of Ivan Matveevich Vinogradov (1891–1983) is also well represented. Delone discusses Vinogradov's work in determining the number of integer points in an arbitrary planar region, estimating Weyl sums, Goldbach's conjecture (Vinogradov proved that all sufficiently large odd numbers are the sum of three odd primes), and Waring's problem.

In 1770 Edward Waring put forth the conjecture that for any positive integer k there exists a value $g(k)$ such that every positive integer is the sum of at most $g(k)$ kth powers (for example, every positive integer is the sum of at most four squares). In 1909, Hilbert proved that $g(k)$ exists for all k. Hardy and Littlewood defined a related quantity $G(k)$ which is the minimum number such that all but finitely many positive integers are the sum of at most $G(k)$ kth powers. For example, $g(3) = 19$, whereas Davenport showed that $G(3) = 16$.

Vinogradov's studied Waring's problem in a series of ten papers between 1924 and 1937, culminating in the book-length article "A new method in analytic number theory". Using his well-known method to estimate trigonometric sums, Vinogradov found that

$$G(n) < 6n(\ln n + 1) \quad \text{for } n \geqslant 16.$$

Delone actually gives a simpler argument than Vinogradov's, proving the less accurate estimate $G(n) < 6n \ln n + 11$.

When judged as a work of history this book has certain weaknesses. As mentioned above, Delone repeatedly gives a geometric interpretation of the work of Korkin and Zolotarev, Markov, and the early work of Voronoĭ in quadratic forms. But there appears to be little historical evidence to support this interpretation. Delone's claims may in fact be true, but they are not documented. Also, the reader needs to be aware of a certain bias in the writing. The works of Russian mathematicians appear to tower over the landscape, with the efforts of Gauss, Dirichlet, Dedekind, Hermite, and Minkowski appearing as distant flashes of lightning on the horizon. For example, in reference to Vinogradov's research on Waring's problem, Delone states (p. 227): "His method [compared to that of Hardy–Littlewood] may therefore be regarded as coming fully within the tradition of the St. Petersburg school whereby deep results are obtained by simple methods." Nevertheless, these qualifications notwithstanding, this work is an able discussion of some fascinating number theory.

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Pioneering Women in American Mathematics: The Pre-1940 PhD's

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While interest in the history of women in mathematics has grown steadily over the last few decades, much of the available scholarship continues to be either biographical or focused on the most famous women mathematicians. Many works fit both categories, even Massimo Mazzotti's recent *The World of Maria Gaetana Agnesi* [Mazzotti, 2007] and Judith P. Zinsser and Julie Candler Hayes' *Emilie Du Châtelet: Rewriting Enlightenment Philosophy and Science* [Zinsser and Hayes, 2006], each of which additionally aim for analytical and contextual treatments of their subjects.

The book under review is thus especially welcome for the several respects in which it represents an ongoing transition in historiography. Green and LaDuke discuss many women whose names are well known, but they also bring attention to those who have faded into